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A polynomial approach to the invariant factors assignment problem via state feedback and output injection

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Abstract

The characterization of the similarity classes of matrices $A + BF + KC + KWF$ obtained by State Feedback and Output Injection on a system (A, B, C, W) are revisited. New, more straighforward proofs are obtained. The results hold working on arbitrary fields.

Keywords: Linear control systems; State feedback; Output injection; Polynomial matrices; Similarity invariants

1. Introduction

Given a linear time-invariant system $(A, B, C, W) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m} \times \mathbb{F}^{p\times n} \times \mathbb{F}^{p\times m}$, its dynamics can be modified by state feedback or output injection. If we apply both actions simultaneously the resulting matrix of the system is $A + BF + KC + KWF$. We are dealing with the following problem:

Problem 1.1. Let $\mathbb{F}$ be an arbitrary field. Let $(A, B, C, W) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m} \times \mathbb{F}^{p\times n} \times \mathbb{F}^{p\times m}$ be a matrix quadruple. Characterize the existence of matrices $F \in \mathbb{F}^{m\times n}$ and $K \in \mathbb{F}^{n\times p}$ such that $A + BF + KC + KWF$ is in a prescribed similarity class.

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For systems (A, B, C) $(W = 0)$ a solution to this problem has been given by Silva [14] when the underlying field has sufficient number of elements. Roca and Zaballa [10] gave another characterization for arbitrary fields in very close terms but with totally different resources through a rather cumbersome proof. Later, Furtado and Silva [2] gave a solution for quadruples (A, B, C, W) when the underlying field has sufficient number of elements.

In this paper we go back on the problem to provide a constructive, more straightforward proof of the characterization of its solution, and the result holds for arbitrary fields.

We start dealing with the problem for triples of matrices.

Problem 1.2. Let $\mathbb{F}$ be an arbitrary field. Given $(A, B, C) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m} \times \mathbb{F}^{p\times n}$ characterize the similarity classes of $A + BF + KG$ when $F \in \mathbb{F}^{m\times n}$, $K \in \mathbb{F}^{n\times p}$.

Afterwards, we will provide another solution for quadruples (Problem 1.1).

The paper is organized as follows: In Section 2 we introduce the notation and preliminary results. In Section 3 we give a solution to Problem 1.2 for systems without infinite elementary divisors. A solution to Problem 1.2 for regular systems (not having minimal indices) is obtained in Section 4. The main result of the present work is the proof of the sufficiency of this case. The solution to the general case is revisited in Section 5. Finally we will solve Problem 1.1 in Section 6.

2. Notation and preliminary results

All along this paper $\mathbb{F}$ will denote an arbitrary field and $\mathbb{F}[s]$ the ring of polynomials with coefficients in $\mathbb{F}$. We represent by greek letters the elements of $\mathbb{F}[s]$. Given $\alpha, \beta \in \mathbb{F}[s]$, the notation $\alpha \mid \beta$ means that α divides β, and $d(\alpha)$ denotes the degree of α. By $\mathbb{F}^{n\times m}$ and $\mathbb{F}[s]^{n\times m}$ we will denote the sets of $n \times m$ matrices with coefficients in $\mathbb{F}$ and in $\mathbb{F}[s]$, respectively. As usual, two polynomial matrices $A(s), B(s) \in \mathbb{F}[s]^{n\times m}$ are said to be *equivalent* if there are unimodular matrices (i.e. polynomial matrices whose determinants are nonzero constants) $U(s) \in \mathbb{F}[s]^{n\times n}$ and $V(s) \in \mathbb{F}[s]^{m\times m}$ such that $B(s) = U(s)A(s)V(s)$. It is well-known that two polynomial matrices are equivalent if and only if they have the same invariant factors [3]. These are monic polynomials $\gamma_1 \mid \cdots \mid \gamma_n$ which characterize each equivalence class. We agree that $\gamma_i = 1$ whenever $i < 0$ and $\gamma_i = 0$ for $i > \operatorname{rank} A(s)$.

Next we recall the concept of feedback-injection equivalence of matrix quadruples: Two matrix quadruples $(A_1, B_1, C_1, W_1), (A_2, B_2, C_2, W_2) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m} \times \mathbb{F}^{p\times n} \times \mathbb{F}^{p\times m}$ are said to be feedback-injection equivalent if there are matrices $P \in \mathbb{F}^{n\times n}$, $T \in F^{p\times p}$, $Q \in F^{m\times m}$, $R \in \mathbb{F}^{n\times p}$ and $S \in \mathbb{F}^{m\times n}$ such that P, Q and T are invertible and

$$\begin{bmatrix} P & R \\ 0 & T \end{bmatrix} \begin{bmatrix} A_1 & B_1 \\ C_1 & W_1 \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ S & Q \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & W_2 \end{bmatrix}. \tag{1}$$

If $W_1 = W_2 = 0$, this relationship yields the feedback-injection equivalence of matrix triples.

As shown in [10], to solve Problem 1.1 (1.2) we can replace the given matrix quadruple (triple) by any other one in its feedback-injection equivalence class.

A complete system of invariants for the feedback-injection equivalence of matrix quadruples is given by the column and row minimal indices, invariant factors and infinite elementary divisors of their characteristic singular pencil (see [4,5])

$$H(s) = \begin{bmatrix} sI_n - A & -B \\ -C & -W \end{bmatrix}.$$

Or, using the terminology of [6,7], the I_1, I_2, I_3, I_4 Morse's lists.

Given (A, B, C, W) and its characteristic pencil $H(s)$, some (perhaps all) invariant factors of $H(s)$ may be trivial (i.e. equal to 1). If $\gamma_1 \mid \cdots \mid \gamma_h$ are its nontrivial invariant factors and $d(\gamma_1) + \cdots + d(\gamma_h) = u$ then $u \geqslant h$ and $\alpha_1 = \cdots = \alpha_{u-h} = 1$, $\alpha_{u-h+1} = \gamma_1, \ldots, \alpha_u = \gamma_h$ are said to be the invariant factors of (A, B, C, W). We also call row minimal indices, column minimal indices and infinite elementary divisors of (A, B, C, W) to those of $H(s)$. We will refer to all of them as a whole as the Kronecker invariants of (A, B, C, W). In fact, if $\alpha_1 \mid \cdots \mid \alpha_u, f_1 \geqslant \cdots \geqslant f_s > f_{s+1} = \cdots = f_{p-(t+v)} = 0$, $\quad c_1 \geqslant \cdots \geqslant c_r > c_{r+1} = \cdots = c_{m-(t+v)} = 0$, $e_1 \geqslant \cdots \geqslant e_t > e_{t+1} = \cdots = e_{t+v} = 1$, are the invariant factors, the row minimal indices, the column minimal indices and the exponents of the infinite elementary divisors of (A, B, C, W) respectively, then it is feedback-injection equivalent to (A_c, B_c, C_c, W_c) where

$$\begin{bmatrix} A_c & B_c \\ C_c & W_c \end{bmatrix} = \left[\begin{array}{cccc|cccc} N & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & R & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & S & 0 & \widehat{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & D & 0 & E & 0 & 0 \\ \hline 0 & \widehat{G} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_v & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \tag{2}$$

with $N \in \mathbb{F}^{u \times u}$ being the natural canonical form (or first normal form, see [3]) whose invariant factors are $\alpha_1 \mid \cdots \mid \alpha_u$ and

$$R = \operatorname{Diag}\{R_1, \ldots, R_s\}, \quad R_i = \begin{bmatrix} 0 & 0 \\ I_{f_i - 1} & 0 \end{bmatrix} \in \mathbb{F}^{f_i \times f_i}, \quad 1 \leqslant i \leqslant s,$$

$$S = \operatorname{Diag}\{S_1, \ldots, S_r\}, \quad S_i = \begin{bmatrix} 0 & I_{c_i - 1} \\ 0 & 0 \end{bmatrix} \in \mathbb{F}^{c_i \times c_i}, \quad 1 \leqslant i \leqslant r,$$

$$D = \operatorname{Diag}\{D_1, \ldots, D_t\}, \quad D_i \begin{bmatrix} 0 & 0 \\ I_{e_i - 2} & 0 \end{bmatrix} \in \mathbb{F}^{(e_i - 1) \times (e_i - 1)}, \quad 1 \leqslant i \leqslant t,$$

$$\widehat{E} = \operatorname{Diag}\{\widehat{E}_1, \ldots, \widehat{E}_r\}, \quad \widehat{E}_i = [0 \ \cdots \ 0 \ 1]^{\mathrm{T}} \in \mathbb{F}^{c_i \times 1}, \quad 1 \leqslant i \leqslant r,$$

$$E = \operatorname{Diag}\{E_1, \ldots, E_t\}, \quad E_i = [1 \ 0 \ \cdots \ 0]^{\mathrm{T}} \in \mathbb{F}^{(e_i - 1) \times 1}, \quad 1 \leqslant i \leqslant t,$$

$$\widehat{G} = \mathrm{Diag}\{\widehat{G}_1, \ldots, \widehat{G}_s\}, \quad \widehat{G}_i = [0 \ \cdots \ 0 \ 1] \in \mathbb{F}^{1\times f_i}, \ 1 \leqslant i \leqslant s,$$
$$G = \mathrm{Diag}\{G_1, \ldots, G_t\}, \quad G_i = [0 \ \cdots \ 0 \ 1] \in \mathbb{F}^{1\times (e_i-1)}, \ 1 \leqslant i \leqslant t.$$

The quadruple (A_c, B_c, C_c, W_c) will be said to be the *Kronecker canonical form* of (A, B, G, W). It can be obtained when the underlying field is an arbitrary field (see [1,9]). The notation displayed here to represent the Kronecker canonical form of a matrix quadruple will be used throughout the paper.

It is worth noticing that $v = \mathrm{rank}\, W$, therefore the degrees of the infinite elementary divisors of any triple (A, B, C) are greater than 1.

For notational simplicity we will assume in the sequel that $\begin{bmatrix} B \\ W \end{bmatrix}$ and $[C \ \ W]$ are of full column and row rank respectively. This means that $m = r + t + v$ and $p = s + t + u$; i.e. there are no minimal indices equal to 0. In fact, the zero minimal indices do not produce any result on the state matrix of the system under state feedback and output injection. We will also denote by $k_i = e_i - 1$, $i = 1, \ldots, t + v$, and by $n_1 = f_1 + \cdots + f_s$, $n_2 = c_1 + \cdots + c_r$ and $n_3 = k_1 + \cdots + k_t$; therefore, $u + n_1 + n_2 + n_3 = n$.

For pairs of matrices the feedback-injection relation (1) reduces to the feedback equivalence of pairs: $(A_1, B_1), (A_2, B_2) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$ are said to be *feedback equivalent* if there are matrices $P \in \mathbb{F}^{n\times n}$, $Q \in \mathbb{F}^{m\times m}$ and $L \in \mathbb{F}^{m\times n}$, P and Q invertible, such that

$$P[A_1 \quad B_1]\begin{bmatrix} P^{-1} & 0 \\ L & Q \end{bmatrix} = [A_2 \quad B_2].$$

If in addition $L = 0$ and $Q = I_m$, then (A_1, B_1) and (A_2, B_2) are said to be *similar or system similar*.

Taking into account that the characteristic singular pencil of a matrix pair $(A, B) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$, $[sI_n - A \ B]$, has neither row minimal indices nor infinite elementary divisors, a complete system of invariants for the feedback equivalence relation reduces to the column minimal indices (also called controllability indices) and the invariant factors.

Moreover, for square matrices, (1) reduces to similarity: two matrices $A, B \in \mathbb{F}^{n\times n}$ are *similar* if and only if $sI_n - A$, $sI_n - B \in \mathbb{F}[s]^{n\times n}$ are equivalent. The invariant factors of A are those of $sI_n - A$ and so, A and B are similar if and only if they have the same invariant factors.

By $\mathscr{C}(A, B) = [B \ \ AB \ \ \cdots \ \ A^{n-1}B] \in \mathbb{F}^{n\times nm}$ we will denote the controlla-bility matrix of (A, B), and (A, B) is said to be controllable if $\mathscr{C}(A, B)$ is a full row rank matrix. An alternative characterization of controllability is that (A, B) is controllable if and only if all the invariant factors of $[sI - A - B]$ are trivial [8].

We will see below that the solution to Problem 1.2 when the triple does not have infinite elementary divisors can be obtained as a consequence of the extension of Rosenbrock's theorem to noncontrollable pairs [15]. It is not hard to see that this result can be stated in the following terms:

Lemma 2.1. *Let $(A, B) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$ be a matrix pair. Let $\alpha_1 \mid \cdots \mid \alpha_u$ be its invariant factors and $l_1 \geqslant \cdots \geqslant l_r > 0 = l_{r+1} = \cdots = l_m$ its controllability indices. Let $\overline{\gamma}_1 \mid \cdots \mid \overline{\gamma}_n$ be given monic polynomials. Then, there exists a matrix $F \in \mathbb{F}^{m\times n}$ such that $A + BF$ has $\overline{\gamma}_1, \ldots, \overline{\gamma}_n$ as invariant factors if and only if*

$$\overline{\gamma}_i = 1, \quad i = 1, \ldots, n-u-r,$$

and if we denote by $\gamma_i = \overline{\gamma}_{n-u-r+i}$, $i = 1, \ldots, u + r$, then

$$\gamma_i \mid \alpha_i \mid \gamma_{i+r}, \quad i = 1, \ldots u,$$

$$(l_1, \ldots, l_r) \prec (d(\delta_r), \ldots, d(\delta_1)),$$

$$\delta_j = \frac{\prod_{i=1}^{u+j} \operatorname{lcm}(\alpha_{i-j}, \gamma_i)}{\prod_{i=1}^{u+j-1} \operatorname{lcm}(\alpha_{i-j+1}, \gamma_i)}, \quad j = 1, \ldots r.$$

Another previous key result to achieve the main result of this paper is the following:

Lemma 2.2 [12]. *Let $sI_n - N \in \mathbb{F}^{n\times n}$ and $G(s) \in \mathbb{F}[s]^{(n+1)\times(n+1)}$. Let $\alpha_1 \mid \cdots \mid \alpha_n$ and $\gamma_1 \mid \cdots \mid \gamma_{n+1}$ be its invariant factors, respectively. Assume that $\sum_{i=1}^{n} d(\gamma_i) = n + d$ with $d > 0$ an integer. Then there exist matrices $k \in \mathbb{F}^{n\times 1}$, $f \in \mathbb{F}^{1\times n}$ and a monic polynomial $p(s)$ of degree d such that $G(s)$ is equivalent to*

$$C(s) = \begin{bmatrix} sI_n - N & k \\ f & p(s) \end{bmatrix}$$

if and only if

$$\gamma_i \mid \alpha_i \mid \gamma_{i+2}, \quad i = 1, \ldots, n.$$

Finally, to find the solution for quadruples we will need the following lemma.

Lemma 2.3 [13,16]. *Let $A \in \mathbb{F}^{n\times n}$. Let $\alpha_1 \mid \cdots \mid \alpha_n$ be its invariant factors. Let $\tau_1 \mid \cdots \mid \tau_n$ be monic polynomials such that $\sum_{i=1}^{n} d(\tau_i) = n$. Let m be a positive integer. Then, there exists a matrix $P \in \mathbb{F}^{n\times n}$ such that* rank $P \leqslant m$ *and $A + P$ has $\tau_1, \ldots, \tau_n$ as invariant factors if and only if*

$$\tau_{i-m} \mid \alpha_i \mid \tau_{i+m}, \quad i = 1, \ldots, n.$$

If we know that some of the invariant factors of the matrix A are trivial, this result can be stated as follows.

Lemma 2.4. *Let $A \in \mathbb{F}^{n\times n}$. Let $\bar{\alpha}_1 \mid \cdots \mid \bar{\alpha}_n$ be its invariant factors. Assume that $\bar{\alpha}_i = 1$, $i = 1, \ldots, n-k$. Let $\bar{\tau}_1 \mid \cdots \mid \bar{\tau}_n$ be monic polynomials such that $\sum_{i=1}^{n} d(\bar{\tau}_i) = n$. Let m be a positive integer. Then, there exists a matrix $P \in \mathbb{F}^{n\times n}$ such that* rank $P \leqslant m$ *and $A + P$ has $\bar{\tau}_1, \ldots, \bar{\tau}_n$ as invariant factors if and only if*

$$\bar{\tau}_i = 1, \quad i = 1, \dots, n-k-m,$$

and if we denote by $\bar{\tau}_i = \bar{\tau}_{n-k-m+i}$, $i = 1, \dots, k+m$ *and by* $\alpha_i = \bar{\alpha}_{n-k+i}$, $i = 1, \dots, k$, *then*

$$\tau_i \mid \alpha_i \mid \tau_{i+2m}, \quad i = 1, \dots k.$$

As mentioned, to solve Problem 1.2 we can assume (A, B, C) to be in Kronecker canonical form and perform state feedback of matrix F and output injection of matrix K on it. Partitioning appropriately F and K, solving Problem 1.2 is then equivalent to finding matrices F_{ij}, K_{ij} such that

$$\begin{bmatrix} N & K_{11}\widehat{G} & 0 & K_{12}G \\ 0 & R + K_{21}\widehat{G} & 0 & K_{22}G \\ \widehat{E}F_{11} & \widehat{E}F_{12} + K_{31}\widehat{G} & S + \widehat{E}F_{13} & \widehat{E}F_{14} + K_{32}G \\ EF_{21} & EF_{22} + K_{41}\widehat{G} & EF_{23} & D + EF_{24} + K_{42}G \end{bmatrix}$$

has prescribed invariant factors.

It was shown in [10, Lemma 3.4] that the terms $\widehat{E}F_{14}$ and $K_{41}\widehat{G}$ in the above matrix can be made zero without changing the structure of the matrix. To achieve it, Lemmas 3.2 and 3.3 of [10] were needed. The same technique and [10, Lemma 3.2] allows us to make zero also the term $K_{31}\widehat{G}$. Hence, solving Problem 1.2 is equivalent to finding matrices F_{ij}, K_{ij} such that

$$\begin{bmatrix} N & K_{11}\widehat{G} & 0 & K_{12}G \\ 0 & R + K_{21}\widehat{G} & 0 & K_{22}G \\ \widehat{E}F_{11} & \widehat{E}F_{12} & S + \widehat{E}F_{13} & K_{32}G \\ EF_{21} & EF_{22} & EF_{23} & D + K_{42}G + EF_{24} \end{bmatrix}$$

has prescribed invariant factors.

We solve first the case when the triple does not have infinite structure.

3. System without infinite elementary divisors

The proof of the solution to Problem 1.2 given in [14] when the underlying field $\mathbb{F}$ has enough number of elements is still true for arbitrary fields when the system (A, B, C) does not have infinite elementary divisors. It is not like that for general triples.

It happens that the solution to Problem 1.2 when the system does not have infinite elementary divisors can be obtained as a consequence of the extension of Rosenbrock's Theorem to noncontrollable pairs (Lemma 2.1). We find it worth pointing out this fact and include the result next.

Theorem 3.1. *Let* $(A, B, C) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m} \times \mathbb{F}^{p\times n}$ *be a matrix triple having* $\alpha_1 \mid \cdots \mid \alpha_u$ *as invariant factors,* $f_1 \geqslant \cdots \geqslant f_s > 0$ *as row minimal indices,* $c_1 \geqslant \cdots \geqslant c_r > 0$ *as column minimal indices and without infinite elementary divi-*

sors. Let $\overline{\gamma}_1 \mid \cdots \mid \overline{\gamma}_n$ *be monic polynomials. Then there exist matrices* $F \in \mathbb{F}^{m\times n}$ *and* $K \in \mathbb{F}^{n\times p}$ *such that* $A + BF + KG$ *has* $\overline{\gamma}_1, \ldots, \overline{\gamma}_n$ *as invariant factors if and only if*

$$\overline{\gamma}_i = 1, \quad i = 1, \ldots, n - (u + s + r), \tag{3}$$

and if we denote by $\gamma_i = \overline{\gamma}_{n-(u+s+r)+i}$, $i = 1, \ldots, u + s + r$, *then there exist monic polynomials* $\mu_1 \mid \cdots |_{\mu_{u+s}}$ *satisfying*

$$\operatorname{lcm}(\alpha_{i-s}, \gamma_i) \mid \mu_i \mid \gcd(\alpha_i, \gamma_{i+r}), \quad i = 1, \ldots, u + s, \tag{4}$$

$$(f_1, \ldots, f_s) \prec (d(\varDelta_s), \ldots, d(\varDelta_1)), \tag{5}$$

$$\varDelta_j = \frac{\prod_{i=1}^{u+j} \operatorname{lcm}(\alpha_{i-j}, \mu_i)}{\prod_{i=1}^{u+j-1} \operatorname{lcm}(\alpha_{i-j+1}, \mu_i)}, \quad j = 1, \ldots, s,$$

$$(c_i, \ldots, c_\tau) \prec (d(\Sigma_r), \ldots, d(\Sigma_1)), \tag{6}$$

$$\Sigma_j = \frac{\prod_{i=1}^{u+s+j} \operatorname{lcm}(\mu_{i-j}, \gamma_i)}{\prod_{i=1}^{u+s+j-1} \operatorname{lcm}(\mu_{i-j+1}, \gamma_i)}, \quad j = 1, \ldots, r.$$

Proof. Notice that we are assuming in this case that $p = s$ and $m = r$.

As it has been said, we are looking for necessary and sufficient conditions for the existence of matrices K_{11}, K_{21}, F_{11}, F_{12} and F_{13} such that

$$\begin{bmatrix} N & K_{11}\widehat{G} & 0 \\ 0 & R + K_{21}\widehat{G} & 0 \\ \widehat{E}F_{11} & \widehat{E}F_{12} & S + \widehat{E}F_{13} \end{bmatrix} \tag{7}$$

has $\overline{\gamma}, \ldots, \overline{\gamma}_n$ as invariant factors.

Let $\bar{\mu}_1 \mid \cdots \mid \bar{\mu}_{u+n_1}$ be the invariant factors of the submatrix

$$N_1 = \begin{bmatrix} N & K_{11}\widehat{G} \\ 0 & R + K_{21}\widehat{G} \end{bmatrix}.$$

We can write

$$N_1^{\mathrm{T}} = \begin{bmatrix} N^{\mathrm{T}} & 0 \\ 0 & R^{\mathrm{T}} \end{bmatrix} + \begin{bmatrix} 0 \\ \widehat{G}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} K_{11}^{\mathrm{T}} & K_{21}^{\mathrm{T}} \end{bmatrix}.$$

Then, by Lemma 2.1, the following conditions are satisfied:

$$\bar{\mu}_i = 1, \quad i = 1, \ldots, n_1 - s, \tag{8}$$

and defining $\mu_i = \bar{\mu}_{i+n_1-s}$ for $i = 1, \ldots, u + s$,

$$\mu_i \mid \alpha_i \mid \mu_{i+s}, \quad i = 1, \ldots, u, \tag{9}$$

$$(f_1, \ldots, f_s) \prec (d(\varDelta_s), \ldots, d(\varDelta_1)),$$

$$\Delta_j = \frac{\prod_{i=1}^{u+j} \operatorname{lcm}(\alpha_{i-j}, \mu_i)}{\prod_{i=1}^{u+j-1} \operatorname{lcm}(\alpha_{i-j+1}, \mu_i)}, \quad 1 \leqslant j \leqslant s.$$

Hence, condition (5) is satisfied.

If $F_1 = [F_{11} \quad F_{12}]$, matrix (7) can be written as

$$\begin{bmatrix} N_1 & 0 \\ \widehat{E}F_1 & S + \widehat{E}F_{13} \end{bmatrix} = \begin{bmatrix} N_1 & 0 \\ 0 & S \end{bmatrix} + \begin{bmatrix} 0 \\ \widehat{E} \end{bmatrix} \begin{bmatrix} F_1 & F_{13} \end{bmatrix}.$$

Applying again Lemma 2.1, the following conditions are satisfied:

$$\overline{\gamma}_i = 1, \quad i = 1, \ldots, n_2 - r \tag{10}$$

and taking $\widehat{\gamma}_i = \overline{\gamma}_{i+n_2-r}$, $i = 1, \ldots, u + n_1 + r$, then

$$\widehat{\gamma}_i \mid \bar{\mu}_i \mid \widehat{\gamma}_{i+r}, \quad i = 1, \ldots, u + n_1, \tag{11}$$

$$(c_1, \ldots, c_r) \prec (d(\sigma_r), \ldots, d(\sigma_1)), \tag{12}$$

$$\sigma_j = \frac{\prod_{i=1}^{u+n_1+j} \operatorname{lcm}(\bar{\mu}_{i-j}, \widehat{\gamma}_i)}{\prod_{i=1}^{u+n_1+j-1} \operatorname{lcm}(\bar{\mu}_{i-j+1}, \widehat{\gamma}_i)}, \quad j = 1, \ldots, r.$$

From conditions (8), (10) and (11), we obtain (3). Moreover, as $\widehat{\gamma}_i = \gamma_{i-(n_1-s)}$, $i = 1, \ldots, u + n_1 + r$,

$$\begin{aligned} \prod_{i=1}^{u+n_1+j} \operatorname{lcm}(\bar{\mu}_{i-j}, \widehat{\gamma}_i) &= \prod_{i=1}^{n_1-s} \widehat{\gamma}_i \prod_{i=n_1-s+1}^{u+n_1+j} \operatorname{lcm}(\mu_{i-j}, \widehat{\gamma}_i) \\ &= \prod_{i=1}^{n_1-s} \gamma_{i-(n_1-s)} \prod_{i=1}^{u+s+j} \operatorname{lcm}(\mu_{i-j}, \gamma_i) = \prod_{i=1}^{u+s+j} \operatorname{lcm}(\mu_{i-j}, \gamma_i), \end{aligned}$$

for $j = 1, \ldots, r$. From this equalities we conclude that if Σ_j is defined as in the statement of the theorem, then $\sigma_j = \Sigma_j$, for $j = 1, \ldots, r$ and (12) is (6).

Finally, (11) can be written as

$$\gamma_i \mid \mu_i \mid \gamma_{i+r}, \quad i = 1, \ldots, u + s.$$

From this condition and (9), condition (4) follows.

Conversely, assume that conditions (3)–(6) are satisfied. From (4) we derive condition (9). From this condition and (5) and taking $\bar{\mu}_i = \mu_{i-(n_i-s)}$, $i = 1, \ldots, u + n_1$, by Lemma 2.1 there exists a feedback matrix $K^{\mathrm{T}} = [K_{11}^{\mathrm{T}}, K_{21}^{\mathrm{T}}]$ such that

$$\begin{bmatrix} N & K_{11}\widehat{G} \\ 0 & R + K_{21}\widehat{G} \end{bmatrix}$$

has $\bar{\mu}_i, \ldots, \bar{\mu}_{u+n_1}$ as invariant factors.

Let us consider now the polynomials $\overline{\gamma}_1 \mid \cdots \mid \overline{\gamma}_n$ of the statement of the theorem. Note that, by (3), $\overline{\gamma}_i = 1$, $i = 1, \ldots, n - (u + n_1) - r$. Let $\overline{\gamma}_i = \overline{\gamma}_{n-(u+n_1)-r+i}$,

$i = 1, \ldots, u + n_1 + r$. It is easy to see from the definitions of the polynomials that $\widehat{\gamma}_i = \gamma_{i-(n_1-s)}$, $i = 1, \ldots, u + n_1 + r$. Therefore, by (4), condition (11) is satisfied. Defining σ_j, $j = 1, \ldots, r$ as in (12), it is possible to prove that $\sigma_j = \Sigma_j$, $j = 1, \ldots, r$ as we did in the necessity part of the proof. As a consequence and using again Lemma 2.1, there exists a feedback matrix $F = [F_1, F_2]$ such that

$$\begin{bmatrix} N_1 & 0 \\ 0 & S \end{bmatrix} + \begin{bmatrix} 0 \\ \widehat{E} \end{bmatrix} \begin{bmatrix} F_1 & F_2 \end{bmatrix} = \begin{bmatrix} N & K_{11}\widehat{G} & 0 \\ 0 & R + K_{21}\widehat{G} & 0 \\ \widehat{E}F_{11} & \widehat{E}F_{12} & S + \widehat{E}F_2 \end{bmatrix}$$

has $\overline{\gamma}_1, \ldots, \overline{\gamma}_{u+n_1+n_2}$ as invariant factors, where $F_1 = [F_{11} \ \ F_{12}]$. $\square$

Remark. It is not hard to check that conditions of [10, Lemma 3.1] and those of this theorem are equivalent. And they are also equivalent to those given in [14].

4. System without minimal indices

In [10] we gave a solution to Problem 1.2 which holds for arbitrary fields. To achieve it we had to reduce the problem to the case when the triple does not have minimal indices. We then characterized the feedback equivalence classes of pairs $(A + KC, B)$ obtained after performing an output injection of matrix K for triples without minimal indices. Finally, we performed a state feedback of matrix F on this type of pairs.

Now, we provide another proof which does not need the characterization of the feedback equivalence classes of the pairs $(A + KC, B)$ as an intermediate step.

For the particular case of regular triples the solution to Problem 1.2 can be written as

Theorem 4.1. *Let $(A, B, C) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m} \times \mathbb{F}^{p\times n}$ be a matrix triple with $\alpha_1 \mid \cdots \mid \alpha_u$ as invariant factors and $k_1 + 1 \geqslant \cdots \geqslant k_t + 1 \geqslant 2$ as degrees of the infinite elementary divisors. Let $\overline{\gamma}_1 \mid \cdots \mid \overline{\gamma}_n$ be monic polynomials. Then there exist matrices $F \in \mathbb{F}^{m\times n}$ and $K \in \mathbb{F}^{n\times p}$ such that $A + BF + KC$ has $\overline{\gamma}_1, \ldots, \overline{\gamma}_n$ as invariant factors if and only if*

$$\sum_{i=1}^{n} d(\overline{\gamma}_i) = n, \tag{13}$$

$$\overline{\gamma}_i = 1, \quad i = 1, \ldots, n - (u + t), \tag{14}$$

and taking $\gamma_i = \overline{\gamma}_{n-(u+t)+i}$, $i = 1, \ldots, u + t$,

$$\gamma_i \mid \alpha_i \mid \gamma_{i+2t}, \quad i = 1, \ldots, u. \tag{15}$$

Proof. The necessity follows from [10, Theorem 4.1]. Let us prove the sufficiency in a more straightforward way than that of [10], using a polynomial approach.

Assume that conditions (13)–(15) are satisfied. Define the polynomials

$$\beta_i^j = \operatorname{lcm}(\alpha_{i-2j}, \gamma_i), \quad i = 1, \ldots, u + j, \quad j = 0, \ldots, t.$$

They satisfy (see [11]) that $\beta_1^j \mid \cdots \mid \beta_{u+j}^j$ and

$$\beta_i^j \mid \beta_i^{j-1} \mid \beta_{i+2}^j, \quad i = 1, \ldots, u + j, \quad j = 1, \ldots, t, \tag{16}$$

with $\beta_i^0 = \alpha_i$, $i = 1, \ldots, u$ and $\beta_i^t = \gamma_i$, $i = 1, \ldots, u + t$. This polynomials form what is called a minimal polynomial path from the sequence of polynomials $\alpha_1 \mid \cdots \mid \alpha_u$ to the sequence $\gamma_1 \mid \cdots \mid \gamma_{u+t}$ and have the following property: Let $D : [0, t] \to \mathbb{R}$. be the function

$$D(x) = \left(\sum_{i=1}^{u+j} d(\beta_i^j) - \sum_{i=1}^{u+j-1} d(\beta_i^{j-1}) \right) (x - j + 1) + \sum_{i=1}^{u+j-1} d(\beta_i^{j-1}),$$

whenever $x \in [j - 1, j]$ for $j = 1, \ldots, t$. It is called the degree function of the minimal polynomial path. Then $D(x)$ is a convex function (see [11]). Let us put $k_0 = u$ and define another function $g : [0, t] \to \mathbb{R}$ as

$$\begin{aligned} &g(x) = k_0 + k_1 + \cdots + k_{j-1} + k_j(x - j + 1), \\ &\quad x \in [j - 1, j], \quad j = 1, \ldots, t. \end{aligned}$$

It is satisfied that $D(0) = g(0) = u$ and $D(t) = g(t) = u + k_1 + \cdots + k_t = n$. Because of the decreasing order of the sequence $k_1 \geqslant \cdots \geqslant k_t$, the function $g(x)$ is concave. Hence, $D(j) \geqslant g(j)$ for $j = 0, \ldots, t$. Therefore, it is possible to find monic polynomials δ^j such that

$$d(\delta^j) = u + k_1 + \cdots + k_j - \sum_{i=1}^{u+j} d(\beta_i^j).$$

Define

$$\bar{\beta}_i^j = \begin{cases} \beta_i^j, & i = 1, \ldots, u + j - 1, \\ \beta_{u+j}^j \delta^j, & i = u + j. \end{cases}$$

These polynomials satisfy $\bar{\beta}_1^j \mid \cdots \mid \bar{\beta}_{u+j}^j$,

$$\sum_{i=1}^{u+j} d(\bar{\beta}_i^j) = u + k_1 + \cdots + k_j,$$

and as a consequence of (16)

$$\bar{\beta}_i^j \mid \bar{\beta}_i^{j-1} \mid \bar{\beta}_{i+2}^j, \quad i = 1, \ldots, u + j - 1. \tag{17}$$

Let us modify the sequences of polynomials $\bar{\beta}_i^j$, $i = 1, \ldots, u + j$, as follows:

$$\begin{aligned} &\alpha_1^j = 1, \quad i = 1, \ldots, k_1 + \cdots + k_j - j, \\ &\alpha_{k_1+\cdots+k_j-j+i}^j = \bar{\beta}_i^j, \quad i = 1, \ldots, u + j \end{aligned}$$

for $j = 0, \ldots, t$. Observe that $\alpha_1^j \mid \cdots \mid \alpha_{u+k_1+\cdots+k_j}^j$, $j = 0, \ldots, t$.

For $j = 1, \dots, t$ let us consider the subsystem (A_j, B_j, C_j) of (A, B, C) given by

$$\left(\begin{bmatrix} N & & & \\ & D_1 & & \\ & & \ddots & \\ & & & D_j \end{bmatrix}, \begin{bmatrix} 0 & & \\ E_1 & & \\ & \ddots & \\ & & E_j \end{bmatrix}, \begin{bmatrix} 0 & G_1 & & \\ & & \ddots & \\ & & & G_j \end{bmatrix} \right)$$

where the nonspecified blocks are zero. Note that the system $(A_j, B_j, C_j) \in \mathbb{F}^{(u+\sum_{i\leqslant j} k_i)\times(u+\sum_{i\leqslant j} k_i)} \times \mathbb{F}^{(u+\sum_{i\leqslant j} k_i)\times j} \times \mathbb{F}^{j\times(u+\sum_{i\leqslant j} k_i)}$. We are going to prove by induction on $j = 1, \dots, t$ that it is possible to find matrices $K_j \in \mathbb{F}^{(u+\sum_{i\leqslant j} k_i)\times j}$ and $F_j \in \mathbb{F}^{j\times(u+\sum_{i\leqslant j} k_i)}$ such that $A_j + B_jF_j + K_jC_j$ has $\alpha_1^j, \dots, \alpha_{u+k_1+\cdots+k_j}^j$ as invariant factors. Once this property is proven, the resulting matrices $K = K_t$ and $F = F_t$ will satisfy the conclusion of the theorem.

Assume that $j = 1$. Then, condition (17) is

$$\bar{\beta}_i^1 \mid \alpha_i \mid \bar{\beta}_{i+2}^1, \quad i = 1, \dots, u,$$

and $\sum_{i=1}^{u+1} d(\bar{\beta}_i^1) = u + k_1$. By Lemma 2.2, there exist $k_1 \in \mathbb{F}^{u\times 1}$, $f_1 \in \mathbb{F}^{1\times u}$ and a monic polynomial $p_1(s)$ of degree k_1:

$$p_1(s) = s^{k_1} + p_{k_1-1}^1 s^{k_1-1} + \cdots + p_1^1 s + p_0^1$$

such that

$$A_1(s) = \begin{bmatrix} sI_u - N & k_1 \\ f_1 & p_1(s) \end{bmatrix}$$

has $\bar{\beta}_1^1, \dots, \bar{\beta}_{u+1}^1$ as invariant factors. Then, matrix

$$B_1(s) = \begin{bmatrix} sI_u - N & 0 & k_1 \\ 0 & I_{k_1-1} & 0 \\ f_1 & 0 & p_1(s) \end{bmatrix}$$

has $\alpha_1^1, \dots, \alpha_{u+k_1}^1$ as invariant factors. Matrix $\mathrm{Diag}(I_{k_1-1}, p_1(s))$ has $p_1(s)$ as a single nontrivial invariant factor, then it is equivalent to the characteristic matrix of its companion matrix P_1

$$sI_{k_1} - P_1 = sI_{k_1} - \begin{bmatrix} 0 & 0 & \cdots & 0 & -p_0^1 \\ 1 & 0 & \cdots & 0 & -p_1^1 \\ 0 & 1 & \cdots & 0 & -p_2^1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -p_{k_1-2}^1 \\ 0 & 0 & \cdots & 1 & -p_{k_1-1}^1 \end{bmatrix}.$$

In other words, there exist unimodular matrices $U_1(s), V_1(s) \in \mathbb{F}[s]^{k_1\times k_1}$ such that

$$U_1(s)\,\mathrm{Diag}(I_{k_1-1}, p_1(s))V_1(s) = sI_{k_1} - P_1.$$

The transformation is given by

$$U_1(s) = \begin{bmatrix} s & s^2 & \cdots & s^{k_1-1} & 1 \\ -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & -1 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & -s & 0 & \cdots & 0 & 0 & -p_1^1 \\ 0 & 1 & -s & \cdots & 0 & 0 & -p_2^1 \\ 0 & 0 & 1 & \cdots & 0 & 0 & -p_3^1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -s & -p_{k_1-2}^1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -p_{k_1-1}^1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}.$$

It allows us the following transformation on matrix $B_1(s)$:

$$\operatorname{Diag}(I_u, U_1(s))B_1(s)\operatorname{Diag}(I_u, V_1(s)) = \begin{bmatrix} sI_u - N & X_1(s) \\ Y_1(s) & sI_{k_1} - P_1 \end{bmatrix}.$$

In fact, $X_1(s) = [0\ \ k_1]V_1(s) = [0\ \ k_1]$, $Y_1(s) = U_1(s)\begin{bmatrix} 0 \\ f_1 \end{bmatrix} = \begin{bmatrix} f_1 \\ 0 \end{bmatrix}$, therefore the former matrix can be written as $sI_{u+k_1} - N_1$ where N_1 exhibits the following form:

$$\left[\begin{array}{c|cc} N & 0 & * \\ \hline * & 0 & * \\ 0 & I_{k_1-1} & * \end{array}\right].$$

(All along the proof the *'s will denote constant elements.) That is to say, N_1 can be described as $A_1 + B_1F_1 + K_1C_1$ for some matrices F_1, K_1 as desired.

Let $1 < j < t$ and assume that there exist matrices $K_j \in \mathbb{F}^{(u+k_1+\cdots+k_j)\times j}$ and $\mathbb{F}_j \in \mathbb{F}^{j\times(u+k_1+\cdots+k_j)}$ such that matrix $N_j = A_j + B_jF_j + K_jC_j$ has $\alpha_1^j, \ldots, \alpha_{u+k_1+\cdots+k_j}^j$ as invariant factors and it is of the form

$$N_j = \left[\begin{array}{c|cc|cc|c|cc} N & 0 & * & 0 & * & \cdots & 0 & * \\ \hline * & 0 & * & 0 & * & \cdots & 0 & * \\ 0 & I_{k_1-1} & * & 0 & * & \cdots & 0 & * \\ \hline * & * & * & 0 & * & \cdots & 0 & * \\ 0 & 0 & 0 & I_{k_2-1} & * & \cdots & 0 & * \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline * & * & * & * & * & \cdots & 0 & * \\ 0 & 0 & 0 & 0 & 0 & \cdots & I_{k_j-1} & * \end{array}\right].$$

Let us see that the property is true for $j+1$. Define

$$\hat{\beta}_i^{j+1} = \bar{\beta}_{i-(k_1+\cdots+k_j-j)}^{j+1}, \quad i = u + k_1 + \cdots + k_j + 1.$$

Then $\hat{\beta}_1^{j+1} \mid \cdots \mid \hat{\beta}_{u+k_1+\cdots+k_j+1}^{j+1}$,

$$\hat{\beta}_i^{j+1} \mid \alpha_i^j \mid \hat{\beta}_{i+2}^{j+1}, \quad i = 1, \ldots, u + k_1 + \cdots + k_j,$$

and $\sum_{i=1}^{u+k_1+\cdots+k_j+1} d(\hat{\beta}_i^{j+1}) = u + k_1 + \cdots + k_j + k_{j+1}$. Proceeding as in case $j = 1$, by Lemma 2.2, there exist $k_{j+1} \in \mathbb{F}^{(u+k_1+\cdots+k_j)\times 1}$, $f_{j+1} \in \mathbb{F}^{1\times(u+k_1+\cdots+k_j)}$ and a monic polynomial $p_{j+1}(s)$ of degree k_{j+1}

$$p_{j+1}(s) = s^{k_{j+1}} + p_{k_{j+1}-1}^{j+1} s^{k_{j+1}-1} + \cdots + p_1^{j+1} s + p_0^{j+1}$$

such that

$$A_{j+1}(s) = \begin{bmatrix} sI_{u+k_1+\cdots+k_j} - N_j & k_{j+1} \\ f_{j+1} & p_{j+1}(s) \end{bmatrix}$$

has $\hat{\beta}_1^{j+1}, \ldots, \hat{\beta}_{u+k_1+\cdots+k_j+1}^{j}$ as invariant factors. Then, matrix

$$B_{j+1}(s) = \begin{bmatrix} sI_{u+k_1+\cdots+k_j} - N_j & 0 & k_{j+1} \\ 0 & I_{k_{j+1}-1} & 0 \\ f_{j+1} & 0 & p_{j+1}(s) \end{bmatrix}$$

has $\alpha_1^{j+1}, \ldots, \alpha_{u+k_1+\cdots+k_{j+1}}^{j+1}$ as invariant factors. We can find now unimodular matrices $U_{j+1}(s), V_{j+1}(s) \in \mathbb{F}[s]^{k_{j+1}\times k_{j+1}}$ of the same form as $U_1(s)$ and $V_1(S)$, respectively, such that

$$U_{j+1}(s)\,\mathrm{Diag}(I_{k_{j+1}-1}, p_{j+1}(s))V_{j+1}(s) = sI_{k_{j+1}} - P_{j+1},$$

where P_{j+1} is the companion matrix of p_{j+1}. Then

$$\begin{aligned} &\mathrm{Diag}(I_{u+k_1+\cdots+k_j}, U_{j+1}(s))B_{j+1}(s)\,\mathrm{Diag}(I_{u+k_1+\cdots+k_j}, V_{j+1}(s)) \\ &\quad = \begin{bmatrix} sI_{u+k_1+\cdots+k_j} - N_j & X_{j+1} \\ Y_{j+1} & sI_{k_{j+1}} - P_{j+1} \end{bmatrix}, \end{aligned}$$

with $X_{j+1}(s) = [0 \;\; k_{j+1}]V_{j+1}(s) = [0 \;\; k_{j+1}]$ and $Y_{j+1}(s) = U_{j+1}(s)\begin{bmatrix} 0 \\ f_{j+1} \end{bmatrix} = \begin{bmatrix} f_{j+1} \\ 0 \end{bmatrix}$. Therefore, the resulting matrix can be written as $sI_{u+k_1+\cdots+k_{j+1}} - N_{j+1}$ where

$$N_{j+1} = \left[\begin{array}{c|cc} N_j & 0 & * \\ \hline * & 0 & * \\ 0 & I_{k_{j+1}-1} & * \end{array}\right].$$

Observe that N_{j+1} is of the form $A_{j+1} + B_{j+1}F_{j+1} + K_{j+1}C_{j+1}$ for some matrices F_{j+1}, K_{j+1}.

By induction, the property is true for all $j = 1, \dots, t$.

For $j = t$ we obtain a matrix $sI_n - N_t$ with $\overline{\gamma}_1, \dots, \overline{\gamma}_n$ as invariant factors. It is a matter of a simple identification to write

$$N_t = A + BF + KC$$

for some matrices K and F. □

5. General case

A solution to Problem 1.2 for the general case has been given in [10]. Taking advantage of the former results, we state here the result in terms of a bit more simple conditions. The idea of the proof follows step by step that of [10]. Nevertheless, due to the specification of the amount of trivial invariant factors in the prescribed similarity class, some modifications have to be introduced. In consequence, we include next the proof.

Theorem 5.1. *Let $(A, B, C) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m} \times \mathbb{F}^{p\times n}$ be a matrix triple with $\alpha_1 \mid \cdots \mid \alpha_u$ as invariant factors, $f_1 \geqslant \cdots \geqslant f_s > 0$ as row minimal indices, $c_1 \geqslant \cdots \geqslant c_r > 0$ as column minimal indices, $k_1 + 1 \geqslant \cdots \geqslant k_t + 1 \geqslant 2$ as degrees of the infinite elementary divisors and without zero minimal indices. Let $\overline{\gamma}_1 \mid \cdots \mid \overline{\gamma}_n$ be monic polynomials. Then there exist matrices $F \in \mathbb{F}^{m\times n}$ and $K \in \mathbb{F}^{n\times p}$ such that $A + BF + KC$ has $\overline{\gamma}_1, \dots, \overline{\gamma}_n$ as invariant factors if and only if*

$$\sum_{i=1}^{n} d(\overline{\gamma}_i) = n, \tag{18}$$

$$\overline{\gamma}_i = 1, \quad i = 1, \dots, n - (u + s + r + t), \tag{19}$$

and taking $\gamma_i = \overline{\gamma}_{n-(u+s+r+t)+i}$, $i = 1, \dots, u + s + r + t$, there exist monic polynomials $\mu_1 \mid \cdots \mid \mu_{u+s}$ such that

$$\operatorname{lcm}(\alpha_{i-s}, \gamma_i) \mid \mu_i \mid \gcd(\alpha_i, \gamma_{i+r+2t}), \quad i = 1, \dots, u + s, \tag{20}$$

$$(f_1, \dots, f_s) \prec (d(\Delta_s), \dots, d(\Delta_1)), \tag{21}$$

$$\Delta_j = \frac{\prod_{i=1}^{u+j} \operatorname{lcm}(\alpha_{i-j}, \mu_i)}{\prod_{i=1}^{u+j-1} \operatorname{lcm}(\alpha_{i-j+1}, \mu_i)}, \quad j = 1, \dots, s,$$

$$(c_1, \dots, c_r) \prec (q + d(\Sigma_r), d(\Sigma_{r-1}), \dots, d(\Sigma_1)), \tag{22}$$

$$\Sigma_j = \frac{\prod_{i=1}^{u+s+j} \operatorname{lcm}(\mu_{i-j}, \gamma_i)}{\prod_{i=1}^{u+s+j-1} \operatorname{lcm}(\mu_{i-j+1}, \gamma_i)}, \quad j = 1, \dots, r,$$

$$q = n - n_3 - \sum_{i=1}^{u+s+r} d(\operatorname{lcm}(\mu_{i-r}, \gamma_i)) \geqslant 0.$$

Remark. We are understanding that if $s = 0$ or $r = 0$ the corresponding condition (21) or (22) disappears. In particular, in the second case ($r = 0$), it follows from its definition that $q = 0$.

Proof. It is not hard to see that the necessity of the conditions (18)–(22) follows from the necessity of the conditions obtained in [10, Theorem 3.5].

Let us proof the sufficiency. Assume that conditions (18)–(22) are satisfied. Let $(\widehat{A}, \widehat{B}, \widehat{C})$ be the triple

$$\begin{bmatrix} \widehat{A} & \widehat{B} \\ \widehat{C} & 0 \end{bmatrix} = \left[\begin{array}{ccc|c} N & 0 & 0 & 0 \\ 0 & R & 0 & 0 \\ 0 & 0 & S & \widehat{E} \\ \hline 0 & \widehat{G} & S & \widehat{E} \end{array}\right] \in \mathbb{F}^{(u+n_1+n_2+s)\times(u+n_1+n_2+r)}.$$

Its invariants are those of (A, B, C) but the infinite elementary divisors.

Let us define $u + n_1 + n_2$ monic polynomials as follows: Let τ be a monic polynomial of degree q. Define

$$\tau_i = \begin{cases} \operatorname{lcm}(\mu_{i-r}, \gamma_i), & i = 1, \dots, u+s+r-1, \\ \operatorname{lcm}(\mu_{u+s}, \gamma_{u+s+r})\tau, & i = u+s+r, \end{cases}$$

and take $\bar{\tau}_i = \tau_{i-(n_1-s+n_2-r)}$, $i = 1, \dots, u + n_1 + n_2$. They satisfy that $\bar{\tau}_1 \mid \cdots \mid \bar{\tau}_{u+n_1+n_2}$ and $\sum_{i=1}^{u+n_1+n_2} d(\bar{\tau}_i) = u + n_1 + n_2$. From (20) it follows that:

$$\operatorname{lcm}(\alpha_{i-s}, \tau_i) \mid \mu_i \mid \gcd(\alpha_i, \tau_{i+r}), \quad i = 1, \dots, u+s. \tag{23}$$

Notice that in case $r = 0$ we have that $\tau = 1$, $\tau_i = \mu_i$ for $i = 1, \dots, u + s$, and the former condition is also satisfied. If we define

$$\sigma_j = \frac{\prod_{i=1}^{u+s+j} \operatorname{lcm}(\mu_{i-j}, \tau_i)}{\prod_{i=1}^{u+s+j-1} \operatorname{lcm}(\mu_{i-j+1}, \tau_i)}, \quad j = 1, \dots, r,$$

then $\sigma_j = \Sigma_j$, $j = 1, \dots, r-1$ and $\sigma = \Sigma_r \tau$. It means that (22) can be written as

$$(c_1, \dots, c_\tau) \prec (d(\sigma_r), \dots, d(\sigma_1)). \tag{24}$$

From (21), (23), (24) and Theorem 3.1, there exist matrices K_1, K_2, F_1, F_2 and F_3 such that $sI_{u+n_1+n_2} - M$, where

$$M = \begin{bmatrix} N & K_1\widehat{G} & 0 \\ 0 & R + K_2\widehat{G} & 0 \\ \widehat{E}F_1 & \widehat{E}F_2 & S + \widehat{E}F_3 \end{bmatrix}$$

has $\bar{\tau}_1, \dots, \bar{\tau}_{u+n_1+n_2}$ as invarriant factors. Take

$$\widehat{F} = \begin{bmatrix} F_1 & F_2 & F_3 \end{bmatrix}, \quad \widehat{K} = \begin{bmatrix} K_1 \\ K_2 \\ 0 \end{bmatrix}.$$

Let us consider now the system $(\widetilde{A}, \widetilde{B}, \widetilde{C})$ defined as

$$\begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & 0 \end{bmatrix} = \left[\begin{array}{cc|c} M & 0 & 0 \\ 0 & D & E \\ \hline 0 & G & 0 \end{array}\right] \in \mathbb{F}^{(n+t)\times(n+t)}.$$

It has $\bar{\tau}_1, \dots, \bar{\tau}_{u+n_1+n_2}$ as invariant factors and its infinite elementary divisors are those of (A, B, C). Observe that $\overline{\gamma}_i = 1, i = 1, \dots, n-(u+n_1+n_2+t)$ and take $\widehat{\gamma}_i = \overline{\gamma}_{n-(u+n_1+n_2+t)+i}, i = 1, \dots, u+n_1+n_2+t$. Then $\widehat{\gamma}_1 \mid \cdots \mid \widehat{\gamma}_{u+n_1+n_2+t}$. We claim that

$$\widehat{\gamma}_i \mid \bar{\tau}_i \mid \widehat{\gamma}_{i+2t}, \quad i = 1, \dots, u+n_1+n_2.$$

In fact, as $\widehat{\gamma}_i = \gamma_{i-(n_1-s+n_2-r)}$, $i = 1, \dots, u+n_1+n_2$, the former condition is equivalent to

$$\gamma_{i-(n_1-s+n_2-r)} \mid \tau_{i-(n_1-s+n_2-r)} \mid \gamma_{i-(n_1-s+n_2-r)+2t}, \\ i = 1, \dots, u+n_1+n_2,$$

which is also equivalent to

$$\gamma_i \mid \tau_i \mid \gamma_{i+2t}, \quad i = 1, \dots, u+s+r,$$

what is true from (20). By Theorem 4.1, there exist matrices $\widetilde{F} = [\widetilde{F}_1 \ \ \widetilde{F}_2]$ and $\widetilde{K} = \begin{bmatrix} \widetilde{K}_1 \\ \widetilde{K}_2 \end{bmatrix}$, such that

$$\widetilde{A} + \widetilde{B}\widetilde{F} + \widetilde{K}\widetilde{C} = \begin{bmatrix} M & \widetilde{K}_1 G \\ E\widetilde{F}_1 & D + E\widetilde{F}_2 + \widetilde{K}_2 G \end{bmatrix} = A + BF + KC$$

has $\overline{\gamma}_1, \dots, \overline{\gamma}_n$ as invariant factors, where

$$F = \begin{bmatrix} \widehat{F} & 0 \\ \widetilde{F}_1 & \widetilde{F}_2 \end{bmatrix}, \quad K = \begin{bmatrix} \widehat{K} & \widetilde{K} \\ 0 & \widetilde{K}_2 \end{bmatrix}.$$

This proves the theorem. $\square$

6. The problem on quadruples

We give next a solution to Problem 1.1. It was mentioned that we can assume the quadruple to be in Kronecker canonical form and not to have zero minimal indices. The problem is solved first for regular quadruples (without minimal indices). The solution to the general case will be given afterwards.

Theorem 6.1. *Let* $(A, B, C, W) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m} \times \mathbb{F}^{p\times n} \times \mathbb{F}^{p\times p}$ *be a regular quadruple. Let* $\alpha_1 \mid \cdots \mid \alpha_u$ *be its invariant factors and* $k_1 + 1 \geqslant \cdots \geqslant k_t + 1 > k_{t+1} + 1 = \cdots = k_{t+v} + 1 = 1$ *the degrees of its infinite elementary divisors. Let* $\overline{\gamma}_1 \mid \cdots \mid$

$\overline{\gamma}_n$ be monic polynomials. Then there exist matrices $F \in \mathbb{F}^{(t+v)\times n}$ and $K \in \mathbb{F}^{n\times(t+v)}$ such that $A + BF + KC + KWF$ has $\overline{\gamma}_1, \ldots, \overline{\gamma}_n$ as invariant factors if and only if

$$\sum_{i=1}^{n} d(\overline{\gamma}_i) = n, \tag{25}$$

$$\overline{\gamma}_i = 1, \quad i = 1, \ldots, n - (u + t + v), \tag{26}$$

and taking $\overline{\gamma}_1 = \overline{\gamma}_{n-(u+t+v)+i}$, $i = 1, \ldots, u + t + v$, then

$$\gamma_i \mid \alpha_i \mid \gamma_{i+2t+2v}, \quad i = 1, \ldots, u. \tag{27}$$

Proof. First of all, notice that if $v = 0$ the result stated in this theorem is that of Theorem 4.1. We have to prove then the theorem for the case $v > 0$.

Assume that there exist matrices $F \in \mathbb{F}^{(t+v)\times n}$ and $K \in \mathbb{F}^{n\times(t+v)}$ such that $A + BF + KC + KWF$ has $\overline{\gamma}_1, \ldots, \overline{\gamma}_n$ as invariant factors.

Denote by $(\widehat{A}, \widehat{B}, \widehat{C})$ the triple in Kronecker canonical form whose invariants are those of the quadruple (A, B, C, W) but the infinite elementary divisors of exponent 1:

$$\begin{bmatrix} \widehat{A} & \widehat{B} \\ \widehat{C} & 0 \end{bmatrix} = \left[\begin{array}{cc|c} N & 0 & 0 \\ 0 & D & E \\ \hline 0 & G & 0 \end{array}\right].$$

Let $\bar{\tau}_1 \mid \cdots \mid \bar{\tau}_n$ be the invariant factors of $A + BF + KC$. Observe that if $F = [F_{ij}]_{i,j=1,2}$ and $K = [K_{ij}]_{i,j=1,2}$ then

$$\begin{aligned} &A + BF + KC \\ &\quad = \begin{bmatrix} N & 0 \\ 0 & D \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ E & 0 \end{bmatrix}\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} + \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}\begin{bmatrix} 0 & G \\ 0 & 0 \end{bmatrix} \\ &\quad = \begin{bmatrix} N & 0 \\ 0 & D \end{bmatrix} + \begin{bmatrix} 0 \\ E \end{bmatrix}\begin{bmatrix} F_{11} & F_{12} \end{bmatrix} + \begin{bmatrix} K_{11} \\ K_{21} \end{bmatrix}\begin{bmatrix} 0 & G \end{bmatrix} \\ &\quad = \widehat{A} + \widehat{B}\widehat{F} + \widehat{K}\widehat{C}, \end{aligned}$$

where

$$\widehat{F} = [F_{11} \quad F_{12}], \quad \widehat{K} = \begin{bmatrix} K_{11} \\ K_{21} \end{bmatrix}.$$

By Theorem 4.1

$$\sum_{i=1}^{n} d(\bar{\tau}_i) = n,$$

$$\bar{\tau}_i = 1, \quad i = 1, \ldots, n - (u + t),$$

and taking $\tau_i = \bar{\tau}_{n-u-t+i}$, $i = 1, \ldots, u + t$,

$$\tau_i \mid \alpha_i \mid \tau_{i+2t}, \quad i = 1, \ldots, u. \tag{28}$$

As rank $(KWF) \leqslant v$, by Lemma 2.4

$$\overline{\gamma}_i = 1, \quad i = 1, \ldots, n - (u + t + v), \tag{29}$$

and taking $\gamma_i = \overline{\gamma}_{n-(u+t+v)+i}$, $i = 1, \ldots, u + t + v$, then

$$\gamma_i \mid \tau_i \mid \gamma_{i+2v}, \quad i = 1, \ldots, u + t. \tag{30}$$

Condition (29) is (26). From (28) and (30) condition (27) follows and the necessity is proven.

Conversely, assume that conditions (25)–(27) are satisfied. We aim to define n monic polynomials $\overline{\pi}_1, \ldots, \overline{\pi}_n$ satisfying

$$\overline{\pi}_1 \mid \cdots \mid \overline{\pi}_n, \tag{31}$$

$$\sum_{i=1}^{n} d(\overline{\pi}_i) = n, \tag{32}$$

$$\overline{\pi}_i = 1, \quad i = 1, \ldots, n - (u + t), \tag{33}$$

and for $\pi_i = \overline{\pi}_{n-(u+t)+i}$, $i = 1, \ldots, u + t$,

$$\pi_i \mid \alpha_i \mid \pi_{i+2t}, \quad i = 1, \ldots, u, \tag{34}$$

$$\gamma_i \mid \pi_i \mid \gamma_{i+2v}, \quad i = 1, \ldots, u + t. \tag{35}$$

Let us consider the minimal polynomial path

$$\beta_i^j = \operatorname{lcm}(\alpha_{i-2j}, \gamma_i), \quad i = 1, \ldots, u + j, \quad j = 0, \ldots, t + v,$$

from the sequence of polynomials $\alpha_1 \mid \cdots \mid \alpha_u$ to the polynomials $\gamma_1 \mid \cdots \mid \gamma_{u+t+v}$. Because of the convexity of its degree function [11]

$$\sum_{i=1}^{u+t} d(\beta_i^t) \leqslant n.$$

Let π be a monic polynomial with $d(\pi) = n - \sum_{i=1}^{u+t} d(\beta_i^t)$ and define

$$\pi_i = \begin{cases} \beta_i^t, & i = 1, \ldots, u + t - 1, \\ \beta_{u+t}^t \pi, & i = u + t. \end{cases}$$

Taking $\overline{\pi}_i = \pi_{i-(n-(u+t))}$, $i = 1, \ldots, n$, it is easy to prove that these polynomials satisfy conditions (31)–(35).

From conditions (31)–(34) and Theorem 4.1 applied to $(\widehat{A}, \widehat{B}, \widehat{C})$, there exist matrices $\widehat{F} \in \mathbb{F}^{t \times n}$ and $\widehat{K} \in \mathbb{F}^{n \times t}$ such that $\widehat{A} + \widehat{B}\widehat{F} + \widehat{K}\widehat{C}$ has $\overline{\pi}_1, \ldots, \overline{\pi}_n$ as invariant factors. Observe that if

$$F_1 = \begin{bmatrix} \widehat{F} \\ 0 \end{bmatrix} \in \mathbb{F}^{(t+v) \times n} \quad \text{and} \quad K_1 = [\widehat{K} \quad 0] \in \mathbb{F}^{n \times (t+v)},$$

then $\widehat{A} + \widehat{B}\widehat{F} + \widehat{K}\widehat{C} = A + BF_1 + K_1C$.

On the other hand, by (25), (26), (33), (35) and Lemma 2.4, there exists a matrix $P \in \mathbb{F}^{n\times n}$ with rank $P \leqslant v$ such that $A + BF_1 + K_1C + P$ has $\overline{\gamma}_i, \dots, \overline{\gamma}_n$ as invariant factors. Let rank $P = v_1 \leqslant v$. Then there exist nonsingular matrices Q, $T \in \mathbb{F}^{n\times n}$ such that

$$Q^{-1}PT^{-1} = \begin{bmatrix} 0_{n-v_1} & 0 \\ 0 & I_{v_1} \end{bmatrix}.$$

We can write matrix P as follows:

$$P = Q\begin{bmatrix} 0_{(n-v)\times t} & 0 & 0 \\ 0 & 0_{v-v_1} & 0 \\ 0 & 0 & I_{v_1} \end{bmatrix}\begin{bmatrix} 0_t & 0 & 0 \\ 0 & I_{v-v_1} & 0 \\ 0 & 0 & I_{v_1} \end{bmatrix}$$
$$\times \begin{bmatrix} 0_{t\times(n-v)} & 0 & 0 \\ 0 & 0_{v-v_1} & 0 \\ 0 & 0 & I_{v_1} \end{bmatrix} T.$$

Denote by

$$K_2 = Q\begin{bmatrix} 0_{(n-v)\times t} & 0 & 0 \\ 0 & 0_{v-v_1} & 0 \\ 0 & 0 & I_{v_1} \end{bmatrix} \in \mathbb{F}^{n\times(t+v)},$$

$$F_2 = \begin{bmatrix} 0_{t\times(n-v)} & 0 & 0 \\ 0 & 0_{v-v_1} & 0 \\ 0 & 0 & I_{v_1} \end{bmatrix} T \in \mathbb{F}^{(t+v)\times n},$$

and $F = F_1 + F_2$, $K = K_1 + K_2$. Then, $A + BF_1 + K_1C + P = A + BF + KC + KWF$ and this matrix has $\overline{\gamma}_i, \dots, \overline{\gamma}_n$ as invariant factors. $\square$

Let us solve the general case.

Theorem 6.2. *Let $(A, B, C, W) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m} \times \mathbb{F}^{p\times n} \times \mathbb{F}^{p\times p}$ be a matrix quadruple with $\alpha_1 \mid \cdots \mid \alpha_u$ as invariant factors, $f_1 \geqslant \cdots \geqslant f_s > 0$ as row minimal indices, $c_1 \geqslant \cdots \geqslant c_r > 0$ as column minimal indices, $k_1 + 1 \geqslant \cdots \geqslant k_t + 1 > k_{t+1} + 1 = \cdots = k_{t+v} + 1 = 1$ as exponents of the infinite elementary divisors and without zero minimal indices. Let $\overline{\gamma}_1 \mid \cdots \mid \overline{\gamma}_n$ be monic polynomials. Then there exist matrices $F \in \mathbb{F}^{m\times n}$ and $K \in \mathbb{F}^{n\times p}$ such that $A + BF + KC + KWF$ has $\overline{\gamma}_1, \dots, \overline{\gamma}_n$ as invariant factors if and only if*

$$\sum_{i=1}^{n} d(\overline{\gamma}_i) = n, \tag{36}$$

$$\overline{\gamma}_i = 1, \quad i = 1, \dots, n - (u + s + r + t + v), \tag{37}$$

and taking $\gamma_i = \overline{\gamma}_{n-(u+s+r+t+v)+i}$, $i = 1, \dots, u + s + r + t + v$, there exists monic polynomials $\mu_1 \mid \cdots \mid \mu_{u+s}$ such that

$$\operatorname{lcm}(\alpha_{i-s}, \gamma_i) \mid \mu_i \mid \gcd(\alpha_i, \gamma_{i+r+2t+2v}), \quad i = 1, \ldots, u+s, \tag{38}$$

$$(f_1, \ldots, f_s) \prec (d(\Delta_s), \ldots, d(\Delta_1)), \tag{39}$$

$$(c_1, \ldots, c_r) \prec (q + d(\Sigma_r), d(\Sigma_{r-1}), \ldots, d(\Sigma_1)), \tag{40}$$

where

$$\Delta_j = \frac{\prod_{i=1}^{u+j} \operatorname{lcm}(\alpha_{i-j}, \mu_i)}{\prod_{i=1}^{u+j-1} \operatorname{lcm}(\alpha_{i-j+1}, \mu_i)}, \quad j = 1, \ldots, s,$$

$$\Sigma_j = \frac{\prod_{i=1}^{u+s+j} \operatorname{lcm}(\mu_{i-j}, \gamma_i)}{\prod_{i=1}^{u+s+j-1} \operatorname{lcm}(\mu_{i-j+1}, \gamma_i)}, \quad j = 1, \ldots, r,$$

$$q = n - n_3 - \sum_{i=1}^{u+s+r} d[\operatorname{lcm}(\mu_{i-r}, \gamma_i)] \geqslant 0.$$

Proof. Assume that the quadruple is in Kronecker canonical form and that there exist matrices F and K such that $A + BF + KC + KWF$ has $\overline{\gamma}_1, \ldots, \overline{\gamma}_n$ as invariant factors. It is then obvious that $\sum_{i=1}^n d(\overline{\gamma}_i) = n$.

Let $\bar{\tau}_1 \mid \cdots \mid \bar{\tau}_n$ be the invariant factors of $A + BF + KC$. Then $\sum_{i=1}^n d(\bar{\tau}_i) = n$. Observe that the Kronecker invariants of the triple (A, B, C) are those of the quadruple but the infinite elementary divisors of exponent 1. By Theorem 5.1

$$\bar{\tau}_i = 1, \quad i = l, \ldots, n - (u+s+r+t), \tag{41}$$

and taking $\tau_i = \bar{\tau}_{n-(u+s+r+t)+i}$, $i = 1, \ldots, u+s+r+t$, then there exist monic polynomials $\mu_1 \mid \cdots \mid \mu_{u+s}$ such that satisfy the following conditions:

$$\operatorname{lcm}(\alpha_{i-s}, \tau_i) \mid \mu_i \mid \gcd(\alpha_i, \tau_{i+r+2t}), \quad i = 1, \ldots, u+s, \tag{42}$$

$$(f_1, \ldots, f_s) \prec (d(\delta_s), \ldots, d(\delta_1)), \tag{43}$$

$$(c_1, \ldots, c_r) \prec (w + d(\sigma_r), d(\sigma_{r-1}), \ldots, d(\sigma_1)), \tag{44}$$

where

$$\delta_j = \frac{\prod_{i=1}^{u+j} \operatorname{lcm}(\alpha_{i-j}, \mu_i)}{\prod_{i=1}^{u+j-1} \operatorname{lcm}(\alpha_{i-j+1}, \mu_i)}, \quad j = 1, \ldots, s,$$

$$\sigma_j = \frac{\prod_{i=1}^{u+s+j} \operatorname{lcm}(\mu_{i-j}, \tau_i)}{\prod_{i=1}^{u+s+j-1} \operatorname{lcm}(\mu_{i-j+1}, \tau_i)}, \quad j = 1, \ldots, r,$$

$$w = n - n_3 - \sum_{i=1}^{u+s+r} d[\operatorname{lcm}(\mu_{i-r}, \gamma_i)] \geqslant 0.$$

As rank $(KWF) \leqslant v$, by Lemma 2.4 we have that

$$\overline{\gamma}_i = 1, \quad i = 1, \ldots, n - (u + s + r + t + v), \tag{45}$$

and taking $\gamma_i = \overline{\gamma}_{n-(u+s+r+t+v)+i}$, $i = 1, \ldots, u + s + r + t + v$, then

$$\gamma_i \mid \tau_i \mid \gamma_{i+2v}, \quad i = 1, \ldots, u + s + r + t. \tag{46}$$

From (46) and (42) we obtain (38). Condition (43) is (39). Moreover, since $\operatorname{lcm}(\mu_{i-r}, \gamma_i) \mid \operatorname{lcm}(\mu_{i-r}, \tau_i)$, $i = 1, \ldots, u + s + r$, then $g \geqslant w \geqslant 0$. Condition (40) follows from (44) if we realize that

$$\Sigma_1 \cdots \Sigma_j = \frac{\prod_{i=1}^{u+s+j} \operatorname{lcm}(\mu_{i-j}, \gamma_i)}{\prod_{i=1}^{u+s} \mu_i} \,\Bigg|\, \frac{\prod_{i=1}^{u+s+j} \operatorname{lcm}(\mu_{i-j}, \tau_i)}{\prod_{i=1}^{u+s} \mu_i} = \sigma_1 \cdots \sigma_j,$$
$$j = 1, \ldots, r,$$

and $c_1 + \cdots + c_r = d(\Sigma_1 \cdots \Sigma_r) + q$.

Conversely, assume that conditions (36)–(40) are satisfied.

Let $(\widehat{A}, \widehat{B}, \widehat{C})$ be the triple

$$\begin{bmatrix} \widehat{A} & \widehat{B} \\ \widehat{C} & 0 \end{bmatrix} = \left[\begin{array}{ccc|c} N & 0 & 0 & 0 \\ 0 & R & 0 & 0 \\ 0 & 0 & S & \widehat{E} \\ \hline 0 & \widehat{G} & 0 & 0 \end{array}\right].$$

Its invariants are those of the quadruple (A, B, C, W) but the infinite elementary divisors.

Define

$$\pi_i = \operatorname{lcm}(\mu_{i-r}, \gamma_i), \quad i = 1, \ldots, u + s + r - 1,$$

$$\pi_{u+s+r} = \operatorname{lcm}(\mu_{u+s}, \gamma_{u+s+r})\pi, \quad d(\pi) = q,$$

and $\overline{\pi}_i = \pi_{i-(n_1+n_2-(s+r))}$, $i = 1, \ldots, u + n_1 + n_2$. It is immediate to check that

$$\overline{\pi}_1 \mid \cdots \mid \overline{\pi}_{u+n_1+n_2}, \tag{47}$$

$$\sum_{i=1}^{u+n_1+n_2} d(\overline{\pi}_i) = u + n_1 + n_2, \tag{48}$$

$$\operatorname{lcm}(\alpha_{i-s}, \pi_i) \mid \mu_i \mid \gcd(\alpha_i, \pi_{i+r}), \quad i = 1, \ldots, u + s. \tag{49}$$

Let us see that they satisfy:

$$(c_1, \ldots, c_r) \prec \left(d(\overline{\Sigma}_r), d(\overline{\Sigma}_{r-1}) \ldots, d(\overline{\Sigma}_1)\right), \tag{50}$$

where

$$\overline{\Sigma}_j = \frac{\prod_{i=1}^{u+s+j} \operatorname{lcm}(\mu_{i-j}, \pi_i)}{\prod_{i=1}^{u+s+j-1} \operatorname{lcm}(\mu_{i-j+1}, \pi_i)}, \quad j = 1, \ldots, r, \tag{51}$$

Observe that for $j = 1, \dots, r-1$,

$$\overline{\Sigma}_1 \cdots \overline{\Sigma}_j = \frac{\prod_{i=1}^{u+s+j} \operatorname{lcm}(\mu_{i-j}, \pi_i)}{\prod_{i=1}^{u+s} \mu_i} = \frac{\prod_{i=1}^{u+s+j} \operatorname{lcm}(\mu_{i-j}, \mu_{i-r}, \gamma_i)}{\prod_{i=1}^{u+s} \mu_i} = \Sigma_1 \cdots \Sigma_j,$$

and $\overline{\Sigma}_1 \cdots \overline{\Sigma}_r = \Sigma_1 \cdots \Sigma_r \pi$. Hence,

$$d(\overline{\Sigma}_1 \cdots \overline{\Sigma}_j) = d(\Sigma_1 \cdots \Sigma_j), \quad j = 1, \dots, r-1,$$
$$d(\overline{\Sigma}_1 \cdots \overline{\Sigma}_r) = d(\Sigma_1 \cdots \Sigma_r) + q,$$

and (50) is true. By (47)–(50), condition (39) and Theorem 3.1, there exist matrices $\widehat{F} \in \mathbb{F}^{r \times n}$ and $\widehat{K} \in \mathbb{F}^{n \times s}$ such that $M = \widehat{A} + \widehat{B}\widehat{F} + \widehat{K}\widehat{G}$ has $\overline{\pi}_1, \dots, \overline{\pi}_{u+n_1+n_2}$ as invariant factors.

Let us consider now the system

$$\begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{W} \end{bmatrix} = \left[\begin{array}{cc|cc} M & 0 & 0 & 0 \\ 0 & D & E & 0 \\ \hline 0 & G & 0 & 0 \\ 0 & 0 & 0 & I_v \end{array}\right]$$

with $\overline{\pi}_1, \dots, \overline{\pi}_{u+n_1+n_2}$ as invariant factors and $k_1 + 1 \geqslant \cdots \geqslant k_t + 1 > k_{t+1} + 1 = \cdots = k_{t+v} + 1 = 1$ as degrees of the infinite elementary divisors.

From (37) we have that $\overline{\gamma}_i = 1$, $i = 1, \dots, n - (u + n_1 + n_2 + t + v)$ for $u + s + r + t + v \leqslant u + n_1 + n_2 + t + v$. Let

$$\widehat{\gamma}_i = \overline{\gamma}_{n-(u+n_1+n_2+t+v)+i}, \quad i = 1, \dots, u + n_1 + n_2 + t + v.$$

Let us see that

$$\widehat{\gamma}_i \mid \overline{\pi}_i \mid \widehat{\gamma}_{i+2t+2v}, \quad i = 1, \dots, u + n_1 + n_2.$$

As $\widehat{\gamma}_i = \gamma_{i-(n_1+n_2-s-r)}$, $i = 1, \dots, u + n_1 + n_2 + t + v$, the former condition is equivalent to

$$\gamma_{i-(n_1+n_2-s-r)} \mid \pi_{i-(n_1+n_2-s-r)} \mid \gamma_{i-(n_1+n_2-s-r)+2t+2v},$$
$$i = 1, \dots, u + n_1 + n_2 + t + v,$$

which is equivalent to

$$\gamma_i \mid \pi_i \mid \gamma_{i+2t+2v}, \quad i = 1, \dots, u + s + r.$$

But this condition is

$$\gamma_i \mid \pi_i = \operatorname{lcm}(\mu_{i-r}, \gamma_i) \mid \gamma_{i+2t+2v}, \quad i = 1, \dots, u + s + r,$$

and this is true.

By Theorem 6.1 there exist matrices $\widetilde{F} \in \mathbb{F}^{t \times n}$ and $\widetilde{K} \in \mathbb{F}^{n \times t}$ such that $\widetilde{A} + \widetilde{B}\widetilde{F} + \widetilde{K}\widetilde{C} + \widetilde{K}\widetilde{W}\widetilde{F}$ has $\overline{\gamma}_1, \dots, \overline{\gamma}_n$ as invariant factors. It is enough to see that if

$$\widetilde{F}=\begin{bmatrix}\widetilde{F}_{11} & \widetilde{F}_{12}\\ \widetilde{F}_{21} & \widetilde{F}_{22}\end{bmatrix},\quad \widetilde{K}=\begin{bmatrix}\widetilde{K}_{11} & \widetilde{K}_{12}\\ \widetilde{K}_{21} & \widetilde{K}_{22}\end{bmatrix}$$

then

$$\begin{aligned}
&\widetilde{A}+\widetilde{B}\widetilde{F}+\widetilde{K}\widetilde{C}+\widetilde{K}\widetilde{W}\widetilde{F}\\
&\quad=\begin{bmatrix}M & 0\\ 0 & D\end{bmatrix}+\begin{bmatrix}0 & 0\\ E & 0\end{bmatrix}\begin{bmatrix}\widetilde{F}_{11} & \widetilde{F}_{12}\\ \widetilde{F}_{21} & \widetilde{F}_{22}\end{bmatrix}+\begin{bmatrix}\widetilde{K}_{11} & \widetilde{K}_{12}\\ \widetilde{K}_{21} & \widetilde{K}_{22}\end{bmatrix}\begin{bmatrix}0 & G\\ 0 & 0\end{bmatrix}+\widetilde{K}\widetilde{W}\widetilde{F}\\
&\quad=\left[\begin{array}{ccc|c}N & K_{11}\widehat{G} & 0 & \\ 0 & R+K_{21}\widehat{G} & 0 & \widetilde{K}_{11}G\\ \hat{E}F_{11} & \hat{E}F_{12}+K_{31}\widehat{G} & S+\hat{E}F_{13} & \\ \hline \multicolumn{3}{c|}{E\tilde{F}_{11}} & D+E\tilde{F}_{12}+\tilde{K}_{21}G\end{array}\right]+\widetilde{K}\widetilde{W}\widetilde{F}\\
&\quad=A+BF+KC+KWF,
\end{aligned}$$

where

$$\widehat{F}=[F_{11}\quad F_{12}\quad F_{13}],\quad \widehat{K}=\begin{bmatrix}K_{11}\\ K_{21}\\ K_{31}\end{bmatrix},$$

$$\begin{bmatrix}\widehat{F} & 0\\ \widetilde{F}_{11} & \widetilde{F}_{12}\\ \widetilde{F}_{21} & \widetilde{F}_{22}\end{bmatrix},\quad K=\begin{bmatrix}\widehat{K} & \widetilde{K}_{11} & \widetilde{K}_{12}\\ 0 & \widetilde{K}_{21} & \widetilde{K}_{22}\end{bmatrix}$$

and the result is proven. □

7. Conclusions

We have given new proofs to the characterization of the similarity classes of matrices $A+BF+KG+KWF$ obtained by State Feedback and Output Injection on a system (A, B, C, W), for different cases of systems. It has been shown that for systems (A, B, C) without infinite elementary divisors, the result is a consequence of the generalization to noncontrollable pairs of Rosenbrock's Theorem. The solution when (A, B, C) is regular (does not have minimal indices) is obtained from a polynomial approach. The general case for systems (A, B, C) is reduced to the latter. Finally, a solution to the general case for systems (A, B, C, W) has been given which only involves in addition a result on the rank variation of a matrix after an additive perturbation of bounded rank.

The proofs are of interest because of two main reasons. On one hand, working on arbitrary fields is no longer possible to perform a conformal transformation and change the system into another one without infinite elementary divisors as in the standard technique. So that we have to face the existence of infinite structure. On the other hand, although the solutions are characterized in terms of similar conditions to those of previous referred works, the roles of finite and infinite elementary divisors are clearly split, the amount of trivial invariant factors is always established and the polynomial paths involved in the characterizations are the minimal polynomials paths defined between two sets of interlacing polynomials in the sense of [11], what means to be stated, within this approach, in the simplest possible form.

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